Topology and its Applications 36 (1990) 157–171
North-Holland

A METHOD FOR CONSTRUCTING EXAMPLES OF M-EQUIVALENT SPACES

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Received 10 November 1989

Two retractions r_1 and r_2 of a space X are called parallel if $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$. Images of the space X under parallel retractions are called parallel retracts of X. We show that if K_1 and K_2 are parallel retracts of a space X, then the R-quotient spaces X/K_1 and X/K_2 are M-equivalent (i.e., their free topological groups in the sense of Markov are topologically isomorphic). This assertion yields a number of examples of M-equivalent spaces.

AMS (MOS) Subj. Class.: 22A05, 54G20, 54B15

free topological groups R-quotient mappings
M-equivalent spaces

Two Tychonoff spaces X and Y are called M-equivalent if their free topological groups in the sense of Markov (or, equivalently, their free topological groups in the sense of [11, 8]) are topologically isomorphic. The notion of M-equivalence was introduced by Graev in [8] where he constructed the first example of nonhomeomorphic M-equivalent spaces and posed a general problem: What topological properties are preserved by the relation of M-equivalence? (We say that a topological property $\mathcal{P}$ is preserved by M-equivalence if for any pair X, Y of M-equivalent spaces, X has $\mathcal{P}$ iff Y has $\mathcal{P}$.) It is sometimes natural to ask if some property is preserved by M-equivalence within some given class of spaces, that is, if we assume that both X and Y are in this class. Note that M-equivalence of two spaces implies their l-equivalence (two spaces are called l-equivalent if their spaces of real-valued continuous functions with the topologies of pointwise convergence are linearly homeomorphic) [4].

Now a number of topological properties and cardinal invariants are known to be preserved by M-equivalence and some weaker equivalences. Some results of this kind can be found in [2, 3, 5, 8, 9, 16–18, 22]. On the other hand, Graev's example in [8] shows that metrizability, first and second axioms of countability, local compactness and Čech-completeness are not preserved by M-equivalence even within the class of countable spaces. Graev's example was generalized in [21] where it was shown that countability of pseudocharacter and the property "to contain a convergent sequence" are not preserved by M-equivalence within the class of compact spaces. In this paper we give another generalization of Graev's example. The main theorem and some examples in this paper were announced in Russian in [12–15].

We use terminology and, with few evident exceptions, notations assumed in [2, 6]. The symbol $F(X)$ denotes the free topological group over a Tychonoff space X in the sense of Markov. Recall the main properties of $F(X)$ [11] which will be used in the sequel without special references:

(1) $F(X)$ without topology is the free group with the set of generators X. Hence, the elements of $F(X)$ are irreducible words of the form $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n$ are elements of X and $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, n$.

(2) X is a closed subspace of $F(X)$.

(3) If $f: X \to G$ is a continuous mapping of X to a topological group G, then the (unique) homomorphism $f^*: F(X) \to G$ extending f is continuous.

We often use the following form of (3): A homomorphism of $F(X)$ to a topological group is continuous iff its restriction to X is continuous.

1. *R*-quotient mappings and free topological groups

A continuous mapping $p: X \to Y$ is called *R-quotient* [10] if $p(X) = Y$ and a real-valued function φ on Y is continuous iff the composition $\varphi \circ p: X \to \mathbb{R}$ is continuous. The following obvious assertion shows that R-quotience is a proper relativization of the notion of quotient mapping to the category of completely regular spaces.

Proposition 1.1. *Let X, Y be spaces, Z a completely regular space, $p: X \to Y$ an R-quotient mapping. Then a mapping $f: Y \to Z$ is continuous iff the composition $f \circ p: X \to Z$ is continuous.*

Corollary 1.2. *An R-quotient bijection defined on a completely regular space is a homeomorphism.*

Let $p: X \to Y$ be a mapping of a space X onto a set Y. It is shown in [10] that there exists the unique completely regular topology on Y, called the *R-quotient topology*, which makes the mapping p R-quotient. The R-quotient topology can be described as the finest completely regular topology on Y making p continuous. The space Y with this topology is called the *R-quotient space* of X with respect to p; in this situation we shall call p the *natural mapping*. We will be especially interested in the case when p has a unique fiber K containing more than one point. Such mappings will be called *K-trivial* and the R-quotient space of a space X with respect to a K-trivial mapping will be denoted X/K.

Generally, the R-quotient space of a Tychonoff space needs not be Hausdorff even if all fibers of the natural mapping are closed. The situation with K-trivial decompositions is better.

Proposition 1.3. *If X is a Tychonoff space and K is a closed set in X, then the R-quotient space X/K is Tychonoff.*

Proof. It suffices to check that the space $Y = X/K$ is Hausdorff. Take two points $y_1 \neq y_2$ in Y. We may assume without loss of generality that y_2 is not in $p(K)$, where $p: X \to Y$ is the natural mapping. Then the fiber $p^{-1}(y_2)$ is a singleton. Let x_2 be the unique element of $p^{-1}(y_2)$. The set $K' = K \cup p^{-1}(y_1)$ is closed in X and does not contain x_2, hence there exists a continuous function $f: X \to \mathbb{R}$ such that $f(x_2) = 0$ and $f(K') \subset \{1\}$. The function $g: Y \to \mathbb{R}$ such that $f = g \circ p$ exists because f is constant on K and is continuous because p is R-quotient. Clearly, $g(y_1) = 1$ and $g(y_2) = 0$, hence y_1 and y_2 have disjoint neighbourhoods in Y. $\square$

Naturally, each quotient mapping is R-quotient. Therefore, the R-quotient topology and the quotient topology coincide iff the latter is completely regular.

Proposition 1.4. *Let K be a closed set in a completely regular space X. Then the natural mapping $p: X \to X/K$ is quotient (= the R-quotient topology on X/K coincides with the quotient topology) iff K is functionally separated with any closed set in X disjoint with K.*

Proof. If p is quotient, then it is closed. Let F be a closed set in X disjoint with K. Then, using the complete regularity of X/K, one can choose a continuous function $f: X/K \to \mathbb{R}$ such that $f(p(K)) = \{0\}$ and $f(p(F)) = \{1\}$. The function $f \circ p$ separates F and K.

Conversely, assume that K is functionally separated with any closed set in X disjoint with K. Now if y_0 is a point in X/K and Φ is closed in X/K and does not contain y_0, then the sets $F_0 = p^{-1}(y_0)$ and $F_1 = p^{-1}(\Phi)$ are closed and disjoint in X and F_0 is either K or a singleton. In both cases there is a continuous function $g: X \to \mathbb{R}$ such that $g(F_0) = \{0\}$ and $g(F_1) \subset \{1\}$. If F_0 is a singleton we can choose g so that also $g(K) = \{1\}$. Then there exists a function $f: X/K \to \mathbb{R}$ such that $g = f \circ p$. The function f is continuous because p is R-quotient and f separates y_0 and Φ. $\square$

Corollary 1.5. *If X is normal and K is closed in X or X is completely regular and K is compact, then the quotient topology and the R-quotient topology on X/K coincide.*

In what follows all considered topological spaces are assumed to be Tychonoff.

Proposition 1.6. *If $p: X \to Y$ is an R-quotient mapping, U is an open subset of Y and $V = p^{-1}(U)$, then the restriction $p \mid V: V \to U$ is R-quotient.*

Proof. Assume that a function $f: X \to Y$ is such that the composition $f \circ p: V \to \mathbb{R}$ is continuous. We prove the continuity of f at an arbitrary point $y_0 \in U$.

Choose a continuous function $g: Y \to \mathbb{R}$ such that $y_0 \in U_1 = \operatorname{Int} g^{-1}(1)$ and $Y \setminus U \subset U_0 = \operatorname{Int} g^{-1}(0)$. Define a function $f_1: Y \to \mathbb{R}$ by the rule: $f_1(y) = f(y)g(y)$ if $y \in U$ and $f_1(y) = 0$ if $y \in Y \setminus U$. The composition $f_1 \circ p$ is continuous on X because it is obviously continuous on V and constant on $p^{-1}(U_0)$, and $\{V, p^{-1}(U_0)\}$ is an open covering of X. Hence, f_1 is continuous on Y and the restriction $f|U_1 = f_1|U_1$ is continuous on the neighbourhood U_1 of y_0. $\square$

A restriction of an R-quotient mapping to a preimage of a closed subspace needs not be R-quotient.

Combining Proposition 1.6 and Corollary 1.2 we get:

Corollary 1.7. *The natural mapping $p: X \to X/K$ maps homeomorphically $X \setminus K$ onto $(X/K) \setminus (p(K))$.*

Proposition 1.8. *A continuous surjection $p: X \to Y$ is R-quotient if and only if the homomorphism $p^*: F(X) \to F(Y)$ extending p is open.*

Proof. Assume that p is R-quotient. The set $H = \ker p^*$ is a closed normal subgroup of $F(X)$. Put $G = F(X)/H$ the quotient group and let $\pi: F(X) \to G$ be the natural homomorphism. Then there exists the unique isomorphism $i: G \to F(Y)$ such that $p^* = i \circ \pi$. Then i is continuous because p^* is continuous and π is open. We are going to show that the inverse isomorphism $j = i^{-1}$ is also continuous. To that end it suffices to check the continuity of the restriction $j|Y$. But $j \circ p^* = \pi$ and $(j|Y) \circ p = \pi|X$. The mapping p is R-quotient (and G is completely regular), hence continuity of π implies continuity of $j|Y$. Thus, i is a topological isomorphism and $p^* = i \circ \pi$ is open.

Conversely, assume that p^* is open. Let $f: Y \to \mathbb{R}$ be a function such that the composition $f \circ p$ is continuous. Then the composition $f^* \circ p^*: F(X) \to \mathbb{R}$, where $f^*: F(Y) \to \mathbb{R}$ is the homomorphism extending f, is continuous. Now openness of p^* implies continuity of f^* and therefore of $f = f^*|Y$. $\square$

We call two mappings $f: X_1 \to Y_1$ and $g: X_2 \to Y_2$ *M-equivalent* if there exist topological isomorphisms $i: F(X_1) \to F(X_2)$ and $j: F(Y_1) \to F(Y_2)$ such that $j \circ f^* = g^* \circ i$, where $f^*: F(X_1) \to F(Y_1)$ and $g^*: F(X_2) \to F(Y_2)$ are homomorphisms extending f and g.

Corollary 1.9. *If mappings f and g are M-equivalent and f is R-quotient, then g is R-quotient.*

Proposition 1.8 yields:

Theorem 1.10. *Let $p_1: X \to Y_1$, $p_2: X \to Y_2$ be R-quotient mappings, $p_1^*: F(X) \to F(Y_1)$, $p_2^*: F(X) \to F(Y_2)$ their homomorphic extensions. If there exists a topological automorphism $i: F(X) \to F(X)$ such that $i(\ker p_1^*) = \ker p_2^*$, then the mappings p_1 and p_2 are M-equivalent.*

Proof. Define a mapping $j: F(Y_1) \to F(Y_2)$ by the formula: $j(a) = p_2^* \circ i(p_1^{*^{-1}}(a))$ for each $a \in F(Y_1)$. One readily checks that j is well defined and is an isomorphism; moreover, $j \circ p_1^* = p_2^* \circ i$. The homomorphism p_1^* is open by Proposition 1.8 and the composition $p_2^* \circ i$ is continuous, therefore j is continuous. Continuity of the inverse isomorphism j^{-1} is checked similarly. □

Note that M-equivalence of two mappings subtends M-equivalence of their ranges.

2. The main theorem

Two retractions r_1, r_2 of a space X are called *parallel* if $r_1 \circ r_2 = r_1$, $r_2 \circ r_1 = r_2$. The term comes from the observation that projections of the plane on two parallel lines are parallel retractions. The images of X under parallel retractions are called *parallel retracts* of X. One easily checks the following assertion.

Proposition 2.1. *Two subsets K_1 and K_2 of a space X are parallel retracts if and only if there is a retraction $r_1: X \to K_1$ mapping homeomorphically K_2 onto K_1.*

Theorem 2.2. *Assume that K_1 and K_2 are parallel retracts of a space X, $Y_1 = X/K_1$ and $Y_2 = X/K_2$ are R-quotient spaces and $p_1: X \to Y_1$ and $p_2: X \to Y_2$ are natural mappings. Then the mappings p_1 and p_2 are M-equivalent. In particular, the spaces Y_1 and Y_2 are M-equivalent.*

Lemma 2.3. *Let K_1, K_2 be subsets of a set X, $p_j: X \to Y_j$ be K_j-trivial mappings and $p_j^*: F(X) \to F(Y_j)$ their homomorphic extensions, $j = 1, 2$. Let, moreover, $i: F(X) \to F(X)$ be an automorphism of $F(X)$ such that $i(K_1) = K_2$. Then $i(\ker p_1^*) = \ker p_2^*$.*

The lemma follows from the fact that $\ker p_j^*$ is the minimal normal subgroup of $F(X)$ containing all words of the form xy^{-1}, $x, y \in K_j$.

Proof of Theorem 2.2. Let $r_1: X \to K_1$, $r_2: X \to K_2$ be parallel retractions. Define a mapping $i_0: X \to F(X)$ by the formula:

$$i_0(x) = r_1(x)x^{-1}r_2(x).$$

Clearly, i_0 is continuous. Let $i: F(X) \to F(X)$ be the homomorphism extending i_0. A straightforward verification using the formulae $r_j \circ r_k = r_j$ for $j, k = 1, 2$ shows that $i(K_1) = K_2$ and that the restriction $i \circ i | X$ is identical. The last assertion implies that $i \circ i$ is the identical automorphism of $F(X)$. Now the theorem follows from Lemma 2.3 and Theorem 1.10. □

The following construction is a source of parallel retracts for constructing examples dealing with cardinal invariants. Let X_0 be a space and K be a retract of X_0, X be the space obtained by adding to X_0 an isolated point (in the sequel the spaces thus obtained will be denoted X_0^+). Put $Y = X_0/K \oplus K$.

Theorem 2.4. *The spaces X and Y defined as above are M-equivalent.*

Proof. Put $Z = X_0 \oplus K_1$, where K_1 is a homeomorphic copy of K. The sets K and K_1 are parallel retracts of Z, hence the natural mappings $p_1 : Z \to Z_1 = Z/K_1$ and $p_2 : Z \to Z_2 = Z/K$ are M-equivalent. To end the proof it remains to notice that Z_1 is homeomorphic to X and Z_2 is homeomorphic to Y. $\square$

We also use the following version of Theorem 2.4. Let $X_1, \ldots, X_n$ be spaces with a point x_i fixed in each X_i, $i = 1, \ldots, n$. A *bunch* of the spaces $X_1, \ldots, X_n$ is the quotient space $(X_1, x_1) \vee \cdots \vee (X_n, x_n) = (X_1 \oplus \cdots \oplus X_n)/\{x_1, \ldots, x_n\}$.

Theorem 2.5. *If K is a retract of X, $x_0 \in K$, then the spaces X and $(X/K, p(K)) \vee (K, x_0)$, where $p : X \to X/K$ is the natural mapping, are M-equivalent.*

Proof. Put $Z = (X, x_0) \vee (K', x_0')$, where K' is a homeomorphic copy of K and x_0' is the point of K corresponding to x_0. Then K and K' are parallel retracts of Z and the spaces Z/K and Z/K' are homeomorphic respectively to $(X/K, p(K)) \vee (K, x_0)$ and X. $\square$

Proposition 2.6. *Let $X_1, \ldots, X_n$ be spaces, x_i and y_i arbitrary points of X_i, $i = 1, \ldots, n$. Then the bunches $(X_1, x_1) \vee \cdots \vee (X_n, x_n)$ and $(X_1, y_1) \vee \cdots \vee (X_n, y_n)$ are M-equivalent.*

Proof. Put $Z = X_1 \oplus \cdots \oplus X_n$. Then the sets $K_1 = \{x_1, \ldots, x_n\}$ and $K_2 = \{y_1, \ldots, y_n\}$ are parallel retracts of Z. By definition, $Z/K_1 = (X_1, x_1) \vee \cdots \vee (X_n, x_n)$, $Z/K_2 = (X_1, y_1) \vee \cdots \vee (X_n, y_n)$ and the spaces Z/K_1 and Z/K_2 are M-equivalent by Theorem 2.2. $\square$

Proposition 2.7. *Assume that K_1 and K_2 are retracts of a space X and there exists a retraction $r_2 : X \to K_2$ such that $r_2(K_1)$ is a singleton. Then the space X is M-equivalent to the bunch $(X/(K_1 \cup K_2), x_0) \vee (K_1, x_1) \vee (K_2, x_2)$ where points $x_0 \in X/(K_1 \cup K_2)$, $x_1 \in K_1$ and $x_2 \in K_2$ are arbitrary.*

Proof. Proposition 2.5 implies M-equivalence of $(X/K_1, p(K_1)) \vee (K_1, x_1)$ and X, where $p : X \to X/K_1$ is the natural mapping. Since r_2 is constant on K_1, there exists a mapping $\bar{r}_2 : X/K_1 \to p(K_2)$ such that $\bar{r}_2 \circ p = p \circ r_2$; $\bar{r}_2$ is continuous because p is R-quotient. Clearly, $\bar{r}_2$ is a retraction. One can extend $\bar{r}_2$ to a retraction $r : (X/K_1, p(K_1)) \vee (K_1, x_1) \to p(K_2)$ by putting $r(x) = \bar{r}_2(p(K_1))$ for all $x \in K_1$. Now

Theorem 2.5 and Proposition 2.6 imply M-equivalence of X and $(X/(K_1 \cup K_2), x_0) \vee (K_1, x_1) \vee (p(K_2), x_2)$ and we are left to check that the restriction $p' = p|K_2 : K_2 \to p(K_2)$ is a homeomorphism. p' is a continuous bijection, hence it suffices to check that p' is R-quotient. Let $f: p(K_2) \to \mathbb{R}$ be a function such that the composition $f_1 = f \circ p'$ is continuous. Then the function $g_1 = f_1 \circ r_2$ is a continuous extension of f_1 over X. Since g_1 is constant on K_1 (recall that $r_2(K_1)$ is a singleton), there exists a function $g: X/K_1 \to \mathbb{R}$ such that $g_1 = g \circ p$. The function g is continuous because p is R-quotient and $f = g|p(K_2)$ is continuous. $\square$

Corollary 2.8. *If $X = A \times B$, then X is M-equivalent to the bunch $(X/K, x_0) \vee (A, a_0) \vee (B, b_0)$, where $K = (A \times \{b_0\}) \cup (\{a_0\} \times B)$ and the points $x_0 \in X/K$, $a_0 \in A$ and $b_0 \in B$ are arbitrary.*

Remark 2.9. As was noted by Čoban, the spaces Y_1 and Y_2 in Theorem 2.2 also have topologically isomorphic free topological objects in any variety of topological universal algebras, the signature of which includes a group operation (free topological rings, free topological modules, etc.).

3. Examples of M-equivalent spaces

Example 3.1. Let D be a discrete space, $I = [0, 1]$ the segment and $X = (I \times D) \oplus D$. Choose a point $d \in D$ and put $K_1 = \{0\} \times D$, $K_2 = D \cup \{(0, d)\}$. Clearly, K_1 and K_2 are parallel retracts of X, hence the R-quotient spaces $Y_1 = X/K_1$ and $Y_2 = X/K_2$ are M-equivalent by Theorem 2.2. The space Y_1 contains $|D|$ many isolated points while Y_2 contains none.

Corollary 3.2. *The cardinality of the set of all isolated points of a space is not preserved by M-equivalence.*

The author does not know the answer to the question raised by Arhangel'skiĭ: Is the property of being a scattered space (= each subspace contains an isolated point) preserved by M-equivalence?

Example 3.3 (obtained jointly with Shakhmatov [15]). Let $\mathbb{Q}$ be the space of rationals and $C = \{x_n: n \in \mathbb{N}^+\} \cup \{x_0\}$ a convergent sequence with the limit point x_0. Define a topology on the product $X_0 = \mathbb{Q} \times C$ by the following conditions: All points (q, x_n) with $q \in \mathbb{Q}$ and $n > 0$ are isolated in X_0 and a set $U \subset X_0$ is a neighbourhood of a point (q, x_0) iff U is a neighbourhood of this point in the product topology on $\mathbb{Q} \times C$. The space X_0 thus defined is countable and metrizable and contains a dense discrete subspace. Therefore, X_0 is pseudocomplete [1] and hence is a Baire space. Clearly, the same remains true for the space $X = X_0^+$.

Put $Y_0 = X_0 \oplus \mathbb{Q}$, $Y = Y_0^+$. Y is not a Baire space because $\mathbb{Q}$ is an open subspace of Y. We are going to show that X and Y are M-equivalent. The set $K_1 = \mathbb{Q} \times \{x_0\}$ is a retract of X_0; by Theorem 2.4, X is M-equivalent to the space $X_1 = (X_0/K_1) \oplus K_1 = (X_0/K_1) \oplus \mathbb{Q}$. Similarly, $K_2 = \mathbb{Q} \times \{x_0\}$ is a retract of Y_0, hence Y is M-equivalent to $Y_1 = (Y_0/K_2) \oplus K_2 = (X_0/K_1) \oplus \mathbb{Q} \oplus \mathbb{Q}$. But the spaces X_1 and Y_1 are homeomorphic because $\mathbb{Q} \oplus \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$.

Corollary 3.4. *The following properties are not preserved by M-equivalence within the class of countable metrizable spaces*:

(a) *the property of being a Baire space,*
(b) *pseudocompleteness,*
(c) *the property "to contain a dense discrete subspace",*
(d) *the property "to contain a dense Čech-complete subspace".*

As we already noted, Čech-completeness and local compactness are not preserved by M-equivalence [8]. Nevertheless, if X and Y are M-equivalent metrizable spaces (in fact, it suffices that Y is a Dieudonné complete space of pointwise countable type), then local compactness of X implies local compactness of Y [9]. The author does not know the answer to the following question raised by Arhangel'skiĭ: Assume that X and Y are second-countable spaces and X is Čech-complete. Is it true then that Y is Čech-complete?

Example 3.5 (obtained jointly with Tkačenko). Let A be a countable space without countable π-base (we can take as A, for example, a countable dense subspace of the Tychonoff cube of weight continuum). Following Example 3.3, define a topology on the product $X_0 = A \times C$, where $C = \{x_n: n \in \mathbb{N}^+\} \cup \{x_0\}$ is a convergent sequence with the limit point x_0, by the conditions: All points (a, x_n) with $a \in A$ and $n > 0$ are isolated in X_0 and a set $U \subset X_0$ is a neighbourhood of a point (a, x_0) in X_0 iff U is a neighbourhood of this point in the product topology on $A \times C$. The space X_0 thus defined is countable and completely regular (see [6]) and contains a dense countable discrete subspace, therefore has a countable π-base. Clearly, the space $X = X_0^+$ also has a countable π-base.

Put $K = A \times \{x_0\}$ and $Y = (X_0/K) \oplus K$. Clearly, K is a retract of X_0, therefore the spaces X and Y are M-equivalent by Theorem 2.4. The space Y contains an open subspace K homeomorphic to A, hence Y has no countable π-base.

Let us make a slight modification of this example. Put $Z_0 = \beta X_0$ and $\bar{K} = \mathrm{Cl}_{Z_0}(K)$. The retraction $r: X_0 \to K$ extends to a retraction $\bar{r}: Z_0 \to \bar{K}$, hence $\bar{K}$ is a retract of Z_0. Put $Z = Z_0^+$ and $Z_1 = (Z_0/\bar{K}) \oplus \bar{K}$. Again, the spaces Z and Z_1 are M-equivalent by Theorem 2.4. The space Z contains a dense countable discrete subspace, therefore has a countable π-base. The space Z_1 contains a closed and open subspace $\bar{K}$ with A dense in $\bar{K}$. Since A has no countable π-base, the spaces $\bar{K}$ and Z_1 cannot have one.

Corollary 3.6. *Countability of the π-weight is not preserved by M-equivalence neither within the class of countable spaces nor within the class of compact spaces.*

Example 3.7. Let $X=\{(x,y)\in\mathbb{R}^2\colon 0<x\leqslant 1,\ y=\sin(1/x) \text{ or } x=0,\ -1\leqslant y\leqslant 1\}$, $K_1=\{(x,y)\in\mathbb{R}^2\colon x=0,\ -1\leqslant y\leqslant 1\}$ and $K_2=\{(x,y)\in\mathbb{R}^2\colon \frac{1}{2}\leqslant x\leqslant 1,\ y=\sin(1/x)\}$ be subspaces of the Eucledian plane $\mathbb{R}^2$. Obviously, K_1 and K_2 are parallel retracts of X, moreover, the quotient space X/K_1 is homeomorphic to the segment $I=[0,1]$ and X/K_2 is homeomorphic to X. Therefore, the spaces X and I are M-equivalent by Theorem 2.2.

Corollary 3.8. *Local connectedness and path connectedness are not preserved by M-equivalence within the class of metrizable continua.*

Note that connectedness is preserved by M-equivalence [22]. In [16] it is shown that M-equivalence preserves homology groups within the class of polyhedra. The next example, observed by Dranišnikov, shows that this does not hold for homotopy groups.

Example 3.9. Let S be a circle, 0 a point of S, $T=S\times S$ a torus. By Corollary 2.8, T is M-equivalent to the bunch $X=T/(S\times\{0\}\cup\{0\}\times S)\vee S\vee S$ which is homeomorphic to the bunch of a two-dimensional sphere with two circles. Now we have $\pi_1(T)=\mathbb{Z}\oplus\mathbb{Z}$ and $\pi_1(X)=\mathbb{Z}*\mathbb{Z}$, where $*$ denotes the free product (and $\mathbb{Z}$ is the group of integers). Furthermore, $\pi_2(T)=0$ and $\pi_2(X)$ is the free Abelian group over an infinite set of generators.

Corollary 3.10. *Homotopy groups are not preserved by M-equivalence within the class of two-dimensional finite polyhedra. In particular, M-equivalent polyhedra need not be homotopically equivalent.*

Dimension dim is preserved by M-equivalence [17]. The author does not know if there exist homotopically equivalent polyhedra of equal dimension which are not M-equivalent.

Example 3.11. Let $X=\{(x,y)\in\mathbb{R}^2\colon 0<x\leqslant 1,\ y=\sin(1/x)\}\cup\{(0,1),\ (0,-1)\}\cup\{(x,2)\in\mathbb{R}^2\colon 0\leqslant x<1\}$ be the subspace of the real plane $\mathbb{R}^2$. Obviously, the sets $K_1=\{(0,1),\ (0,2)\}$ and $K_2=\{(1,\sin 1),\ (0,2)\}$ are parallel retracts of X, therefore the spaces $Y_1=X/K_1$ and $Y_2=X/K_2$ are M-equivalent by Theorem 2.2. One easily checks that the spaces Y_1 and Y_2 are homeomorphic respectively to subspaces $\{(x,y)\colon 0<x\leqslant 1,\ y=\sin(1/x)\}\cup\{(x,1)\colon -2<x\leqslant 0\}\cup\{(0,-1)\}$ and $\{(x,y)\colon x>0,\ y=\sin(1/x)\}\cup\{(0,1),\ (0,-1)\}$ of $\mathbb{R}^2$. The space Y_1 admits a continuous bijection onto a compact subspace $\{(x,y)\colon 0<x\leqslant 1,\ y=\sin(1/x)\}\cup\{(0,y)\colon -1\leqslant y\leqslant 1\}$ of $\mathbb{R}^2$ (the semisegment $\{(x,1)\colon -2<x\leqslant 0\}$ turns over the point $(0,1)$ becoming part of the segment $\{(0,y)\colon -1\leqslant y\leqslant 1\}$). We are going to show that Y_2 does not admit a continuous bijection onto a compact space.

It is convenient to consider Y_2 as an extension of the real line $\mathbb{R}$ with the remainder $Y_2 \setminus \mathbb{R}$ consisting of two points $y_1 = (0, 1)$ and $y_2 = (0, -1)$. The only property of this extension essential for the following argument is that both y_1 and y_2 are in the closure of each left ray $(-\infty, a)$ of the real line $\mathbb{R}$.

Now assume that there exists a continuous bijection h of Y_2 onto a compact space K. Since Y_2 is connected, K is a continuum. Choose disjoint closed neighbourhoods U_1 and U_2 of points $h(y_1)$ and $h(y_2)$ in K. Put $F_i = h^{-1}(U_i)$, $i = 1, 2$. Then F_1 and F_2 are closed disjoint neighbourhoods of y_1 and y_2 in Y_2. Clearly, neither F_1 nor F_2 can contain a whole left ray of $\mathbb{R}$. As F_1 and F_2 are disjoint, one of them, say F_1, contains no whole right ray $(a, +\infty)$ of $\mathbb{R}$. Then F_1 contains no ray of $\mathbb{R}$ and one can choose a two-sided sequence of reals $\{a_k : k \in \mathbb{Z}\}$ such that $a_k \in \mathbb{R} \setminus F_1$ and $a_{k+1} > a_k$ for all $k \in \mathbb{Z}$, $\lim_{k \to +\infty} a_k = +\infty$ and $\lim_{k \to -\infty} a_k = -\infty$. Then the sets $\Phi_k = [a_k, a_{k+1}] \cap F_1$, $k \in \mathbb{Z}$, form a compact covering of the set $F_1 \cap \mathbb{R}$. Let C be a component of the point $h(y_1)$ in U_1. Then $C = \bigcup \{h(\Phi_k \cap h^{-1}(C)) : k \in \mathbb{Z}\} \cup \{h(y_1)\}$. C is a continuum and $\Phi_k \cap h^{-1}(C)$ are compact, therefore, by Sierpiński's theorem [19], $C = \{h(y_1)\}$. Then $\{h(y_1)\}$ is a component of a closed neighbourhood U_1 of $h(y_1)$, which contradicts [6, Lemma 6.1.25] claiming that $C \cap \operatorname{Fr} U_1$ cannot be empty. The contradiction thus obtained shows the impossibility of a continuous bijection of Y_2 onto a compact space.

Corollary 3.12. *The property "to admit a continuous bijection onto a compact space" is not preserved by M-equivalence within the class of connected one-dimensional σ-compact second-countable spaces.*

The author does not know whether the property "to admit a continuous bijection onto a σ-compact space" is preserved by M-equivalence.

The hereditary Lindelöf property is preserved by M-equivalence [18]; in particular, a compact space which is M-equivalent to a perfectly normal compact is itself perfectly normal. The next example shows that the situation with hereditary normality is different.

Example 3.13. Let $A = D \cup \{a\}$ be the one-point compactification of an uncountable discrete space D, $X = A \times A$. The space $X \setminus \{(a, a)\}$ is nonnormal and not Dieudonné complete, in particular, nonparacompact (see [6]). By Corollary 2.8, the space X is M-equivalent to the space

$$Y = (X/K, y_0) \vee (A \times \{a\}, (a, a)) \vee (\{a\} \times A, (a, a)),$$

where $K = (A \times \{a\}) \cup (\{a\} \times A)$ and $y_0 = p(a, a)$, where $p : X \to X/K$ is the natural mapping. The space Y is compact and has no nonisolated points except point $p(a, a)$. Therefore, Y is homeomorphic to A. Clearly, A is hereditarily paracompact. Note that A is a scattered Eberlein compact space, therefore X also is.

Corollary 3.14. *Hereditary paracompactness, hereditary normality and hereditary Dieudonné completeness are not preserved by M-equivalence within the class of scattered Eberlein compacta.*

A space X is Dieudonné complete iff the free Abelian topological group over X is complete [20], therefore the Dieudonné completeness is preserved by M-equivalence. The question whether paracompactness is preserved by M-equivalence [4] remains open. The next example shows that normality is not preserved by M-equivalence.

Example 3.15. Let T be the space of all ordinals $\leqslant \omega_1$ with the order topology, $T_0 = T\backslash\{\omega_1\}$, $X_0 = T \times T\backslash\{(\omega_1, \omega_1)\}$, $K = T_0 \times \{\omega_1\} \subset X_0$. The mapping $r: X_0 \to K$ defined by

$$r(\alpha, \beta) = (\min(\alpha, \beta), \omega_1)$$

is a continuous retraction. Thus, K is a retract of X_0. By Theorem 2.4, the space $X = X_0^+$ is M-equivalent to $Y = Y_0 \oplus K$, where $Y_0 = X_0/K$.

The space X is not normal because the sets $K = T_0 \times \{\omega_1\}$ and $K = \{\omega_1\} \times T_0$ are closed in X and are not functionally separated in X. We are going to check that the space Y is normal. Since $Y = Y_0 \oplus K$ where K is homeomorphic to a normal space T_0, it suffices to check that Y_0 is compact. Let $p: X_0 \to Y_0 = X_0/K$ be the natural mapping. Consider the continuous extension $\bar{p}: \beta X_0 \to \beta Y_0$. As $T = \beta T_0$, $T_0 \times T_0 \subset X \subset T \times T$ and $T_0 \times T_0$ is countably compact, we have $\beta(T_0 \times T_0) = T \times T$ [7] and $\beta X_0 = T \times T = X_0 \cup \{(\omega_1, \omega_1)\}$. The point (ω_1, ω_1) is in the closure of $T_0 \times \{\omega_1\} = K$. Hence, the point $\bar{p}(\omega_1, \omega_1)$ is in the closure of $\bar{p}(K)$. But $\bar{p}(K) = p(K)$ is a singleton, therefore $\bar{p}(\omega_1, \omega_1)$ is an element of $p(K)$ and consequently of Y_0. Thus, $\bar{p}(\beta X_0) = \bar{p}(X_0) \cup \{\bar{p}(\omega_1, \omega_1)\} = Y_0$ and Y_0 is compact.

Note that the space $Y = Y_0 \oplus K$ is countably compact, hence normality of Y implies its collectionwise normality.

Corollary 3.16. *Normality and collectionwise normality are not preserved by M-equivalence within the class of countably compact spaces.*

It is unknown whether the Lindelöf property is preserved by M-equivalence [4]. The next example shows that countability of extent is not preserved by M-equivalence. Recall that the *extent* of a space X is the supremum of cardinalities of closed discrete subsets of X.

Example 3.17. Let T and T_0 denote the same spaces as in Example 3.15. Put $X_0 = (T_0 \times T) \cup S$ (endowed with the topology of subspace of $T \times T$), where

$$S = \{(\omega_1, \alpha): \alpha < \omega_1, \alpha \text{ is a successor}\}.$$

Put $X = X_0^+$. Clearly, S is an uncountable closed discrete set in X, therefore the extent of X is uncountable.

The set $K = T_0 \times \{\omega_1\}$ is a retract of X_0 (with the retraction given in Example 3.15). By Theorem 2.4, the space X is M-equivalent to $Y = Y_0 \oplus K$, where $Y_0 = X_0/K$. We are going to check that the extent of Y is countable. The space K is countably compact (it is homeomorphic to T_0), so it suffices to check that the extent of Y_0 is countable.

Let $p: X_0 \to Y_0$ be the natural mapping. Since

$$T_0 \times T_0 \subset X_0 \subset T \times T = \beta(T_0 \times T_0),$$

we have:

$$X' = X_0 \cup \{(\omega_1, \omega_1)\} \subset \beta X_0$$

and the mapping p extends to a continuous mapping $\bar{p}: X' \to \beta Y_0$. An argument similar to one in Example 3.15 shows that $\bar{p}(X') = Y_0$. But the space X' contains no closed uncountable discrete sets because it is the union of a countably compact space $T_0 \times T$ and a Lindelöf space $S \cup (\omega_1, \omega_1)$. Now it remains to use the obvious fact that continuous mappings do not raise extent.

Corollary 3.18. *Countability of extent is not preserved by M-equivalence within the class of pseudocompact spaces.*

The last example gives the negative answer to the questions raised in [4]: Are countability of tightness and the k-property preserved by M-equivalence? Recall that a set A is called *bounded* in a space X if each continuous real-valued function on X is bounded on A. A space X is called a *b-space* if for each nonclosed subset F of X there exists a bounded subset A of X such that the intersection $F \cap A$ is not closed in A. Clearly, each k-space is a b-space.

Example 3.19. Let $T = S \cup S_1$ be the Alexandroff double circle (see [6, Example 3.1.26]), where S is homeomorphic to a circle and all points of S_1 are isolated in T. Let $A = \{a_n: n \in \mathbb{N}^+\} \cup \{a_0\}$ be the convergent sequence with the limit point a_0. Put

$$Z = (T \times A) \setminus (S \times \{a_0\}),$$

$X_0 = Z \times Z$ and $X = X_0^+$.

Clearly, X is a locally compact first-countable space. In particular, X is a k-space with countable tightness.

Obviously, the diagonal $\Delta = \{(z, z): z \in Z\}$ is a retract of $X_0 = Z \times Z$. By Theorem 2.4, the space X is M-equivalent to the space $Y = Y_0 \oplus \Delta$, where $Y_0 = X_0 / \Delta$. We are going to prove that Y_0 (and therefore Y) is not a b-space and that the tightness of Y_0 (therefore, of Y) is uncountable.

Put $D = S_1 \times \{a_0\}$. The set D is closed and discrete in Z.

Lemma 3.20. *Let $\{z_n: n \in \mathbb{N}^+\}$ be a sequence of distinct points of D. There exists a continuous function $\varphi: Z \to \mathbb{R}$ such that $\varphi(z_n) = 2^n$ for each $n \in \mathbb{N}^+$.*

Proof. Let, for each $n \in \mathbb{N}^+$, s_n be the point of S_1 such that $z_n = (s_n, a_0)$. Define the function φ on Z by the rule:

$$\varphi(z) = \begin{cases} 2^n & \text{if } z = z_n \text{ or } z = (s_n, a_k) \text{ with } k > n, \\ 0, & \text{otherwise.} \end{cases}$$

Let us check continuity of φ. The family of sets $\{T\times\{a_n\}, \{s\}\times A\colon n\in\mathbb{N}^+, s\in S_1\}$ is an open covering of Z, hence it suffices to check continuity of restrictions of φ on the elements of this covering. But each such restriction is constant except on a finite set of isolated points. $\square$

Put $F=(D\times D)\setminus\Delta$. The set F is closed and discrete in X_0.

Lemma 3.21. *For each infinite countable subset Q of F there exists a continuous real-valued function f on X_0 such that*
(a) $f(\Delta)=\{0\}$,
(b) $f(q)\geqslant 1$ *for each* $q\in Q$ *and*
(c) f *is unbounded on* Q.

Proof. The set $P=\pi_1(Q)\cup\pi_2(Q)$, where $\pi_1, \pi_2\colon Z\times Z\to Z$ are projections, is an infinite countable subset of D. Enumerate elements of P: $P=\{z_n\colon n\in\mathbb{N}^+\}$ and take a function $\varphi\colon Z\to\mathbb{R}$ as in Lemma 3.20. The function f on $X_0=Z\times Z$ defined by

$$f(z_1, z_2)=|\varphi(z_1)-\varphi(z_2)|$$

is as required. $\square$

Lemma 3.22. *For each neighbourhood U of the diagonal Δ in X_0 the intersection $F\cap\mathrm{Cl}_{X_0}(U)$ is nonempty.*

Proof. Fix for each $n\in\mathbb{N}^+$ a neighbourhood V_n of the set $S\times\{a_n\}$ in Z such that $V_n\times V_n\subset U$ (possibility of this follows from the Wallace theorem [6]). Let us consider the set

$$B=\{s\in S_1\colon (s, a_n)\in V_n \text{ for all } n\in\mathbb{N}^+\}.$$

For each $n\in\mathbb{N}^+$ the set $B_n=\{s\in S_1\colon (s, a_n)\notin V_n\}$ is finite because $\{a_n\}\times T$ is compact, therefore the completion $S_1\setminus B=\bigcup\{B_n\colon n\in\mathbb{N}^+\}$ is at most countable. Thus, the set B is uncountable.

Take two distinct points $s_1, s_2\in B$. We are going to show that the point $x_0=(s_1, a_0, s_2, a_0)$ is in the intersection $F\cap\mathrm{Cl}_{X_0}(U)$. Clearly, x_0 is in F. Let us check that x_0 is in the closure of U.

Take arbitrary neighbourhoods G_1 and G_2 of the points (s_1, a_0) and (s_2, a_0). Then the set

$$\{n\in\mathbb{N}^+\colon (s_1, a_n)\notin G_1 \text{ or } (s_2, a_n)\notin G_2\}$$

is finite and we can find an $m\in\mathbb{N}^+$ such that $(s_1, a_m, s_2, a_m)\in G_1\times G_2$. As s_1 and s_2 are in B, we have: $(s_1, a_m)\in V_m$ and $(s_2, a_m)\in V_m$, i.e., $(s_1, a_m, s_2, a_m)\in V_m\times V_m$. Since $V_m\times V_m$ is in U, we proved that each neighbourhood of x_0 meets U and x_0 is in the closure of U. $\square$

Now let $p\colon X_0\to Y_0=X_0/\Delta$ be the natural mapping. Put $L=p(F)$ and $\{d\}=p(\Delta)$. Clearly, $d\notin L$. On the other hand, d is a limit point of L; otherwise $\{d\}$ and L

would be functionally separated and so would be Δ and F, in contradiction with Lemma 2.22. Now let M be an arbitrary infinite countable subset of L. By Lemma 3.21, there exists a continuous function $f: X_0 \to \mathbb{R}$ such that $f(\Delta) = \{0\}$, $f(q) \geqslant 1$ for each $q \in p^{-1}(M)$ and f is unbounded on $p^{-1}(M)$. The mapping p is Δ-trivial and f is constant on Δ, hence there exists a function $g: Y_0 \to \mathbb{R}$ such that $f = g \circ p$. The function g is continuous because p is R-quotient. Clearly, $g(d) = 0$, $g(m) \geqslant 1$ for each $m \in M$ and g is unbounded on M. This means, first, that d is not in the closure of M and, second, that M is unbounded in Y_0. Thus, d is a limit point for no countable subset of L, which means that the tightness of Y_0 is uncountable, and the intersection of a nonclosed set L with each bounded subset of Y_0 is finite, which means that Y_0 is not a b-space.

Corollary 3.23. *There exist M-equivalent spaces X and Y such that X is a first-countable locally compact space and Y is not a b-space and the tightness of Y is uncountable. In particular, the following properties are not preserved by M-equivalence*:

(a) *bisequentiality,*
(b) *the Fréchet-Urysohn property,*
(c) *sequentiality,*
(d) *k-property,*
(e) *b-property,*
(f) *countability of tightness.*

Nonnormality of the space X played an essential role in Example 3.19. The author does not know the answer to the question raised in [5]: Is it true that countability of tightness is preserved by M-equivalence within the class of normal spaces? Note that if X and Y are M-equivalent spaces and Y is a k-space, then sequentiality of X implies sequentiality of Y and the tightness of Y does not exceed the tightness of X [3].

Remark 3.24. Let X_0, Δ and Y be as in Example 3.20, $Z = X_0 \oplus \Delta$ and $p_1: Z \to X = X_0^+$, $p_2: Z \to Y_0 \oplus \Delta$ be the natural mappings as in the proof of Theorem 2.4. By Theorem 2.2, the mappings p_1 and p_2 are M-equivalent. Clearly, p_1 is closed and open while p_2 is not quotient (otherwise Y_0 would be a k-space). Consequently, no property of mappings intermediate between quotience and closed-and-openness is preserved by M-equivalence of mappings.

Question 3.25. Assume that a mapping f is M-equivalent to a perfect mapping. Must f be perfect?

Acknowledgement

I would like to thank Professor A.V. Arhangel'skiĭ for encouraging this work and D.B. Shakhmatov, M.G. Tkačenko, A.N. Dranishnikov and V.V. Tkačuk for valuable comments on it.

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